MINIMIZING THE THICKNESS OF AN INHOMOGENEOUS LAYER AT A GIVEN REFLECTION COEFFICIENT FOR A MONOCHROMATIC WAVE

K. A. Lur'e and M. M. Machevariani

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 10, No. 1, pp. 44–50, 1969

The problem of wave absorption in an inhomogeneous impedance layer is a familiar one in acoustics and electrodynamics. In practice, impedance systems are designed with account for known optimum requirements, the most important of which is the requirement that the absorbing layer be of minimum thickness. The corresponding mathematical problems for sound absorbers in air were treated by various artificial approaches toward the end of the 1930's by Malyuzhinets, who attracted the attention of the authors, and by Svirskii, in a dissertation (Moscow State University, 1943). Similar (and more general) optimization problems can be studied systematically when they are treated as Mayer-Bolza variational problems. That point of view is adopted in this paper, in which the thickness of an inhomogeneous layer on which a plane monochromatic wave is incident normally is minimized.

1. Basic equations and formulation of the optimization problem. The equations for the sound field in an inhomogeneous medium are [1]

$$\frac{\partial p}{\partial t} + \rho c^2 \operatorname{div} \mathbf{v} = 0, \qquad \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\rho} \operatorname{grad} p = 0, \tag{1.1}$$

where p is the sonic pressure, $\mathbf{v}$ is the velocity in the sound wave, and ρ and c are the density of the medium and the sound velocity in it. In general, ρ and c are functions of the coordinates.

Assuming normal incidence of a plane monochromatic wave (a time dependence $e^{-i\omega t}$ is assumed throughout this paper) on an inhomogeneous layer whose properties depend on only one coordinate, we write (1.1) as

$$-i\omega p + \rho c^2 \frac{dv}{dx} = 0, \qquad -i\omega v + \frac{1}{\rho}\frac{dp}{dx} = 0. \tag{1.2}$$

Using

$$g = -v/p \tag{1.3}$$

to replace the oscillatory velocity and pressure by the input admittance g, we easily find from Eqs. (1.2) the following:

$$\frac{dg}{dx} = -i\frac{\omega}{\rho c^2} + i\omega\rho g^2. \tag{1.4}$$

We assume that the medium from which the plane wave is incident on the layer is characterized by an acoustic impedance $\rho_0 c_0$, and that the medium (or system of media) behind the inhomogeneous layer is characterized by an input admittance $g^{(0)}$ (Fig. 1). We further assume that the layer is artificially made inhomogeneous by directional changes in ρ_m and c_m of the initial material. We introduce the notation

$$G = \rho_0 c_0 g, \quad \tau = \frac{\omega}{c_0} x, \qquad \frac{\rho_m}{\rho_0} = \rho^\circ, \qquad k^2 = \frac{c_0^2}{c_m^2}, \tag{1.5}$$

and replace (1.4) by an equation for the dimensionless input admittance:

$$\frac{dG}{d\tau} = -i\frac{k^2}{\rho^\circ} + i\rho^\circ G^2. \tag{1.6}$$

Below, we use the notation

$$k^2 = 1 + (1 + i\eta)\ Q(\tau) \tag{1.7}$$

for the square of the dimensionless propagation constant in the inhomogeneous layer. This method of writing the

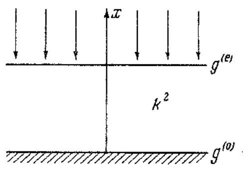

Fig. 1

propagation constant covers many problems of wave propagation in man-made acoustic media [2, 3]. Assuming finally that $\rho^\circ = 1$, i. e., assuming that the inhomogeneity in the layer is due only to a coordinate-dependent propagation constant $k(\tau)$, we replace (1.6) by a system of equations for the real and imaginary parts of the admittance $G = p + iq$:

$$Dp \equiv \frac{dp}{d\tau} - \eta Q + 2pq = 0, \quad Dq \equiv \frac{dq}{d\tau} - p^2 + q^2 + 1 + Q = 0. \tag{1.8}$$

An analogous system of equations can easily be obtained for normal incidence of a plane magnetic wave on an inhomogeneous layer.

We turn now to the formulation of the optimization problem. The phase coordinates p and q, which describe the behavior of the system, satisfy differential equations (1.8). The position of the system at the initial point $\tau = 0$ is given by

$$p(0) = p^{(0)}, \quad q(0) = q^{(0)}. \tag{1.9}$$

We require that the values of the phase coordinates $p(\tau_e)$ and $q(\tau_e)$ at some (not fixed) value $\tau = \tau_e$ be related by

$$\gamma_p \equiv p(\tau_l) - p_l = 0, \quad \gamma_q \equiv q(\tau_l) - q_l = 0. \tag{1.10}$$

Furthermore, we assume

$$\psi \equiv Q(N - Q) - \nu^2 = 0, \tag{1.11}$$

where, in correspondence with the generally accepted terminology, Q and ν are control functions. The function ν is an additional control introduced in order that the restriction on Q contained in the inequality $N \geq Q \geq 0$ (the condition for the physical realization of the inhomogeneous-layer model) could be written as the equivalent equality (1.11). This is a fundamental point, since it permits the reduction of the variational problem with a one-sided extremum to a problem with an arbitrary extremum.

The optimization problem is formulated in the following manner. We are to determine the phase coordinates $p(\tau)$ and $q(\tau)$ which satisfy Eq. (1.8) and the initial conditions (1.9), and the control functions Q and ν related by (1.11), such that the functional

$$J = \tau_e \tag{1.12}$$

has a minimum, when conditions (1.10) hold. Because of the restrictions on the function Q, finite discontinuities are permitted in the phase-coordinates derivatives $dp/d\tau$ and $dq/d\tau$ in the interval $0 \leq \tau \leq \tau_l$. The functions $p(\tau)$ and $q(\tau)$ are assumed continuous over the entire interval.

2. **Necessary conditions for the stationarity of the functional.** As usual, these conditions are obtained from an examination of the first variation of the expression

$$I = J + \int_0^{\tau_l} [\lambda_p Dp + \lambda_q Dq - \mu\psi]\, d\tau + \chi_p\gamma_p + \chi_q\gamma_q, \tag{2.1}$$

in which $\lambda_p(\tau)$, $\lambda_q(\tau)$, $\mu(\tau)$, χ_p, and χ_q are undetermined Lagrange multipliers. The right-hand side of (2.1) differs from J by vanishing, so that the J and I stationarity conditions are the same. In each subinterval between switching points (i. e., between the discontinuities in Q), we have: system (1.8); Eq. (1.11); the equations

$$\frac{d\lambda_p}{d\tau} = 2q\lambda_p - 2p\lambda_q, \qquad \frac{d\lambda_q}{d\tau} = 2p\lambda_p + 2q\lambda_q, \tag{2.2}$$

$$\eta\lambda_p - \lambda_q + \mu(N - 2Q) = 0, \qquad 2\mu\nu = 0; \tag{2.3}$$

the boundary conditions on λ_p and λ_q,

$$\begin{aligned} &\lambda_p(\tau_l) + \frac{\partial}{\partial p(\tau_l)}[J + \chi_p\gamma_p + \chi_q\gamma_q] = 0, \\ &\lambda_q(\tau_l) + \frac{\partial}{\partial q(\tau_l)}[J + \chi_p\gamma_p + \chi_q\gamma_q] = 0; \end{aligned} \tag{2.4}$$

and the equation

$$\frac{d}{d\tau_l}[J+\chi_p\gamma_p+\chi_q\gamma_q]=0. \tag{2.5}$$

At the switching points, the phase coordinates $p(\tau)$ and $q(\tau)$ must be continuous, and the following Erdman-Weierstrass conditions must be satisfied:

$$\lambda_p^-(\tau_M)=\lambda_p^+(\tau_M), \qquad \lambda_q^-(\tau_M)=\lambda_q^+(\tau_M),$$
$$\left[\lambda_p^-\frac{dp^-}{d\tau}+\lambda_q^-\frac{dq^-}{d\tau}-\lambda_p^+\frac{dp^+}{d\tau}-\lambda_q^+\frac{dq^+}{d\tau}\right]_{\tau=\tau_M}=0. \tag{2.6}$$

Eliminating the multipliers χ_p and χ_q from (2.4) and (2.5) with account of (1.10) and (1.12), we find

$$\lambda_p(\tau_e)\frac{dp(\tau_e)}{d\tau_e}+\lambda_q(\tau_e)\frac{dq(\tau_e)}{d\tau_e}=1. \tag{2.7}$$

If the right-hand sides of Eqs. (1.8) and (1.11) do not explicity depend on τ, equality (2.7) holds for any τ, which in this case is the first integral of Eqs. (1.8) and (2.2):

$$H=H_\lambda+H_\mu\equiv\lambda_p[\eta Q-2pq]+$$
$$+\lambda_q[-1-Q+p^2-q^2]+\mu\psi=1. \tag{2.8}$$

3. **Extreme partial arcs.** It is evident from (2.3) that we can satisfy the second equation by either setting $\mu = 0$ (the so-called singular equation) or by setting $\nu = 0$. The singular equation determines those parts of the optimum trajectory in the admittance plane which can be found independently of the restrictions on the control function Q, as is easily seen. It follows from the first equation in (2.3) that, in this case, we have

$$\eta\lambda_p-\lambda_q=0, \tag{3.1}$$

but it then follows from (2.2) that $p(\tau)$ vanishes identically; according to the first equation in (1.8), this means that the function $Q(\tau)$ also vanishes.

The case $\nu = 0$ leads to other parts of the extreme admittance hodograph. It follows from (1.11) that, in this case, $Q(\tau)$ is equal to either 0 or N (the maximum possible value).

Accordingly, the extreme arcs are hodographs corresponding to constant wave numbers. In this case, system (1.8) is easily integrated: we find

$$p+iq=\frac{p_0+iq_0-i\sqrt{1+(1+i\eta)Q}+\operatorname{tg}[\tau\sqrt{1+(1+i\eta)Q}]}{1-i(p_0+iq_0)[\sqrt{1+(1+i\eta)Q}]^{-1}+\operatorname{tg}[\tau\sqrt{1+(1+i\eta)Q}]}. \tag{3.2}$$

Here $Q = 0$, N; (p_0, q_0) is the point on the admittance plane corresponding to the beginning of the τ change in any subinterval within which Q remains constant. It is easy to see that system (2.2) for the Lagrange multipliers λ_p and λ_q can also be integrated in closed form. In this case, however, it is not necessary to obtain explicit expressions for these multipliers, since the minimum layer width τ_l and the optimum admittance hodograph can be found on the basis of p and q alone [Eq. (3.2)]. This is true because there is only one switching for each of the control functions $Q(\tau)$ in the regions $\Phi \gtrless 0$ (see below). To prove this latter assertion, we will proceed as in the theory of optimum systems [4].

4. **Sequence of extreme partial arcs.** We rewrite the first integral of (2.8) as

$$H_\lambda\equiv KQ-2pq\lambda_p+(p^2-q^2-1)\lambda_q=1, \tag{4.1}$$

where

$$K=\frac{\partial H_\lambda}{\partial Q}=\eta\lambda_p-\lambda_q. \tag{4.2}$$

Since both the phase coordinates p and q and the Lagrange multipliers λ_p and λ_q must be continuous at the switching point, we conclude from (4.1) that the first integral is constant at the switching point only if the "switching" function K vanishes at this point (since Q changes discontinuously here). It follows from (4.2) for function K and from the principle of the maximum that the functional H_λ reaches a maximum at Q = N with optimum control functions if K > 0 and at Q = 0 if K < 0. We can thus determine from the sign of the switching function whether the optimum partial arc corresponds to Q = 0 or Q = N. From the behavior of the switching function near the switching point we can determine whether a switching occurs from Q = 0 to Q = N, or vice versa. For this purpose, using (4.2) and (2.2) and the expression for the first integral in the form (4.1), we obtain the following differential equation for the switching function:

$$\frac{1}{2}\frac{dK}{d\tau} = \left[q - \frac{p}{\eta} - \frac{p\,(1+\eta^2)\,(Q - 2\eta^{-1}pq)}{\Phi}\right] K + \frac{(1+\eta^2)\,p}{\Phi}, \tag{4.3}$$

where

$$\Phi = 2pq + \eta\,(1 + p^2 - q^2). \tag{4.4}$$

Representing the function K by a Taylor series near the switching point, and taking into account (2.6) and (2.7), we obtain from this equation the following conditions for the optimum arcs with maximum and vanishing Q:

$$\frac{(1+\eta^2)\,p}{\Phi}\,(\tau - \tau_M) \gtrless 0. \tag{4.5}$$

On physical grounds, we see that p cannot be negative; we therefore conclude from (4.5) that only Q transitions from 0 to N are possible where $\Phi > 0$ on the admittance plane and only from 0 to N where $\Phi < 0$. It follows that the extreme admittance hodograph contains only one switching point in each region: from 0 to N in the $\Phi > 0$ region and from N to 0 in the $\Phi < 0$ region (if, of course, the phase point returns to the region on the admittance plane at which its motion began, a second switching may occur in this region; but again conditions (4.5) hold). The intersection of the extreme admittance hodographs and the boundary of these regions occurs in the following manner. From the condition for the boundedness of the switching function and from Eq. (4.3), we find an expression for K at point on the hyperbola $\Phi = 0$:

$$K = \frac{1}{Q - 2\eta^{-1}pq}. \tag{4.6}$$

Using the principal of the maximum, we find from (4.6) that the optimum admittance hodograph corresponding to the solution of system (1.8) at Q = N may intersect the hyperbola only at points at which

$$2pq = \eta\,(-1 + p^2 - q^2) \leqslant \eta N. \tag{4.7}$$

In an analogous manner, we find that the partial arcs of the optimum admittance hodograph for which Q = 0 cross the boundary of the regions $\Phi \gtrless 0$ at

$$2\eta^{-1}\,pq < 0, \text{ i. e., } q > 0. \tag{4.8}$$

Inequalities (4.7) and (4.8) thus show that optimum transitions from the $\Phi > 0$ region to the $\Phi < 0$ region (or vice versa) along admittance hodographs corresponding to the maximum and minimum values of the propagation constant may occur only at specific segments of the boundary of these regions. This is shown in Fig. 2, where these regions are hatched and where we are given the Q values at which the corresponding transitions may occur.

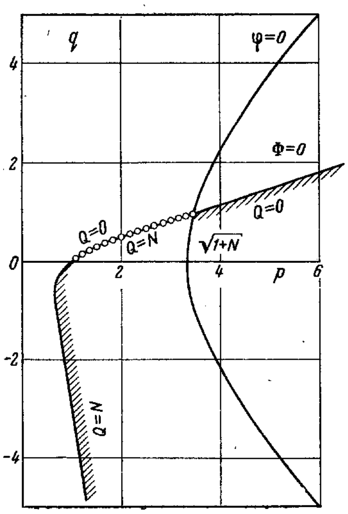

Fig. 2

Finally, we show two examples of constructing optimum inhomogeneous layers corresponding to the extreme hodographs passing through the points (6.6, 7), (1,0) and (2, −1), (1, 0).

In the first case, the admittance hodograph is completely in the region $\Phi > 0$; in the second, it begins at the point (2, −1) in region $\Phi < 0$, passes across the hyperbola $\Phi = 0$, and ends at the point (1, 0). Both problems are easily solved graphically, as shown in Figs. 3 and 4. We construct a hodograph passing through the point (1, 0) by integrating system (1.8) backwards with Q = N, and we construct the hodograph

$$|\beta| = \text{const} \quad \left(|\beta| = \frac{(1-p)^2 + q^2}{(1+p)^2 + q^2}\right),$$

which passes through the point (6.6, 7). After graphically finding the intersection of these hodographs, we find the unknown thicknesses of the two homogeneous layers according to Eq. (3.2). Along each Q = N curve or Q = 0 curve in Figs. 3 and 4 are scale markers corresponding to the reduced layer thicknesses. The calculations were carried out according to Eq. (3.2) for Q = N and Q = 0.

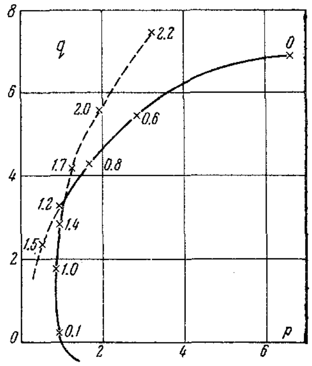

Fig. 3

In the second example, one must construct, in addition to the hodograph beginning at the point (1, 0), and analogous hodograph beginning at (+2, −1), which represents the load admittance. In this case, as is easily seen from Fig. 4, there is a family of extreme hodographs consisting of successive arcs corresponding to solutions of systems (1.8) at Q = N, Q = 0, Q = N. The optimum solution will be that for which the thickness of the inhomogeneous layer, which now consists in general of three homogeneous layers, is at a minimum. This hodograph is found graphically by simply choosing the proper circle $|\beta|$ = const (Fig. 4).

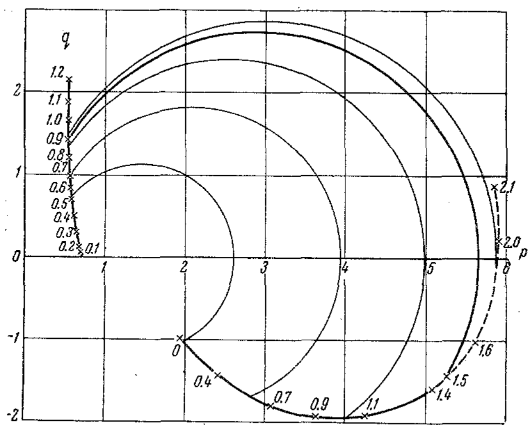

Fig. 4

These problems can, of course, also be solved very effectively by computers. For this purpose, one must integrate the equations of system (1.8) for p and q under conditions (1.9) and Eqs (2.2) for the Lagrange multipliers λ_p and λ_q. The sequences of Q values in Eqs. (1.8) are set up automatically, depending on the behavior of the switching function $K = \eta\lambda_p - \lambda_q$. The greatest difficulty here is that the values for λ_p and λ_q at the beginning of the integration are not known beforehand. They must be determined by trial and error; values must be chosen until integration of systems (1.8) and (2.2) satisfies condition (4.1) and condition (1.9) at the end of the interval. Actually, it

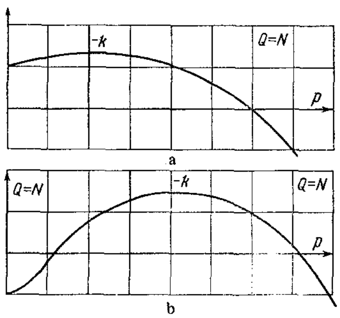

Fig. 5

is necessary only to choose the ratio λ_p/λ_q, since the switching signs (when there are discontinuous changes in the function Q) are determined by the zeros of the switching function, whose positions do not depend on multiplication of this function by an arbitrary constant. Figure 5 illustrates this possibility with the results of a solution of the same two problems on an MN-8 analog computer.

REFERENCES

1. L. M. Brekhovskikh, Waves in Layered Media [in Russian], Izd. AN SSSR, Moscow, 1957.

2. C. Zwikker and C. W. Kosten, Sound Absorbing Materials [Russian translation], Izd-vo inostr. lit., Moscow, 1952.

3. E. Skudrzyk, Fundamentals of Acoustics [Russian translation], Izd. inostr. lit., Moscow, 1959.

4. Optimization Methods with Applications to the Mechanics of Space Flight [Russian translation], izd, Nauka, Moscow, 1965.

1 April 1968

Leningrad